Department of Mathematics
Parahyangan Catholic University
Indonesia

The dynamical system generated by the greedy algorithm

Jonathan Hoseana
Joint work with Steven

Basic idea

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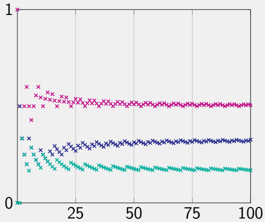
- Density of integers corresponding to each possible residue?



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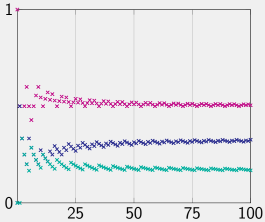

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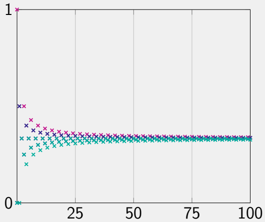

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- Define $\mathbf{G}_A : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$\mathbf{G}_A(x) = x - \mathbf{g}_A(x),$$

where $\mathbf{g}_A(x)$ is the largest element of $\{0, 1, \ldots, \min(A) - 1\} \cup A$ not exceeding x.

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$$x_{n+1} = \mathbf{G}_A(x_n)$$

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for every $n \in \mathbb{N}$.
- Obtain a representation of x_1:

$$x_1 = \mathbf{g}_A(x_1) + \cdots + \mathbf{g}_A\left(x_{R_A(x_1)-1}\right) + r_A(x_1),$$

where

$$R_A(x_1) = \min\{n \in \mathbb{N} : x_n < \min(A)\} \quad \text{and} \quad r_A(x_1) = x_{R_A(x_1)}.$$

A prototypical special case

$$A = \mathbb{P}$$

(primes, including 1 "for convenience")

S. S. Pillai (1930)

https://upload.wikimedia.org/wikipedia/commons/0/05/S.S._Pillai.jpg

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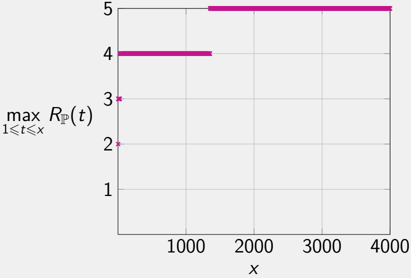

Fact [Pillai, 1930]

We have $\limsup_{x\to\infty} R_{\mathbb{P}}(x) = \infty$.

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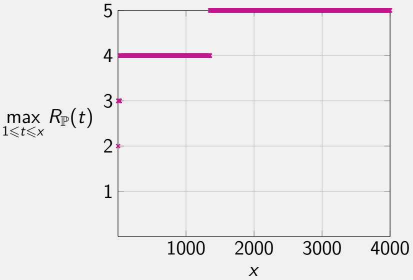

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Proof

Given $x \in \mathbb{N}$. Choose consecutive $p_1, p_2 \in \mathbb{P}$ with $p_2 - p_1 \geqslant x + 1$.
$\left(\prod_{p \leqslant x+1} p + 2, \dots, \prod_{p \leqslant x+1} p + x + 1 \text{ all composite.} \right)$
Then $\mathbf{G}_{\mathbb{P}}(p_1 + x) = x$, and so $R_{\mathbb{P}}(p_1 + x) = R_{\mathbb{P}}(x) + 1$.

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$$\begin{aligned}
\eta_2 &= 1 = 1 + 0,\\
\eta_3 &= 4 = 3 + 1 + 0,\\
\eta_4 &= 27 = 23 + 3 + 1 + 0,\\
\eta_5 &= 1354 = 1327 + 23 + 3 + 1 + 0,\\
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Fact

For every $k \geqslant 2$ we have $\mathbf{G}_{\mathbb{P}}(\eta_{k+1}) = \eta_k$.

Thus for every $k \geqslant 2$ we have $\eta_{k+1} = \eta_k + p_1$, where (p_1, p_2) is the first pair of consecutive primes with $p_2 - p_1 \geqslant \eta_k + 1$.

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Bertrand's postulate [Chebyshev, 1852]

For every integer $x \geqslant 2$ there exists $p \in \mathbb{P}$ such that $x < p < 2x$.

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Consequence

For every $x \in \mathbb{N}_0$ we have

$$1 \leqslant x_{R_{\mathbb{P}}(x)-1} \leqslant \frac{x_{R_{\mathbb{P}}(x)-2}}{2} \leqslant \frac{x_{R_{\mathbb{P}}(x)-3}}{2^2} \leqslant \cdots \leqslant \frac{x_1}{2^{R_{\mathbb{P}}(x)-2}} = \frac{x}{2^{R_{\mathbb{P}}(x)-2}}.$$

Thus,

$$R_{\mathbb{P}}(x) \ll \ln x.$$

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An improvement of Bertrand's postulate [Hoheisel, 1930]

There exist $\theta \in (0, 1)$ and $X_0 \in \mathbb{N}$ such that for every $x \geqslant X_0$ the interval $[x - x^{\theta}, x]$ contains a prime.

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Consequence [Luca & Thangadurai, 2009]

There exist $\theta \in (0,1)$ and $X_0' \in \mathbb{N}$ such that for every $x \geqslant X_0'$ we have

$$X_0' \leqslant x_{K_{\mathbb{P}}(x)} \leqslant x_{K_{\mathbb{P}}(x)-1}{}^{\theta} \leqslant x_{K_{\mathbb{P}}(x)-2}{}^{\theta^2} \leqslant \cdots \leqslant x_1{}^{\theta^{K_{\mathbb{P}}(x)-1}} = x^{\theta^{K_{\mathbb{P}}(x)-1}},$$

where $K_{\mathbb{P}}(x) := \max\{k \in \mathbb{N} : x_k \geqslant X_0'\}$. Thus,

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For every $k \geqslant 2$ we have $\xi_{k+1} = \xi_k + q_1$, where (q_1, q_2) is the first pair of consecutive prime powers with $q_2 - q_1 \geqslant \xi_k + 1$.

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If $|A| = \infty$, say $A = \{a_n\}_{n=1}^{\infty}$ with $a_1 < a_2 < \cdots$, then

- $\limsup_{x\to\infty} R_A(x) = \infty$ if and only if $\limsup_{n\to\infty} (a_n - a_{n-1}) = \infty$;

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  - there exist $X_0 \in \mathbb{N}$ such that for every $x \geqslant X_0$ we have $\mathbf{G}_A(x) \leqslant f(x)$,
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Two notable special cases

$$f(x) = \delta x,\ \delta \in (0,1) \Rightarrow R_A(x) \ll \ln x$$
$$f(x) = x^{\theta},\ \theta \in (0,1) \Rightarrow R_A(x) \ll \ln\ln x$$

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Applications [Mukhopadhyay, et al., 2015]

- Let A be the set of all primes of the form $m^2 + n^2 + 1$ where $m, n \in \mathbb{N}$ and $\gcd(m, n) = 1$. One can take $f(x) = x^{115/121}$ [Wu, 1998]. Thus,

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- Let A be the set of all square-free numbers: those which are divisible by no square other than 1. One can take $f(x) = x^{1/5} \ln x$ [Filaseta and Trifonov, 1992]. Thus,

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$$R_A(x) \ll \ln \ln x.$$
- Let $A = A_{\mathcal{B}}$ be the set of all $\mathcal{B}$-free numbers: those which are divisible by no element of a fixed set $\mathcal{B} = \{b_k\}_{k=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty} 1/b_k < \infty$ and $\gcd(b_i, b_j) = 1$ for all $i \neq j$. One can take $f(x) = x^{33/79}$ [Zhai, 2000]. Thus,
$$R_A(x) \ll \ln \ln x.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$.

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

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Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 0$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{\},$$
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Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 1$ we have

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Suppose $A = \{3, 8, 18, 21\}$. For $x = 2$ we have

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Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 3$ we have

$$r_A^{-1}(\{0\}) \cap [0,x] = \{0,3\},$$
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$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x\to\infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0,x]\right|}{x+1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 4$ we have

$$r_A^{-1}(\{0\}) \cap [0,x] = \{0,3\},$$
$$r_A^{-1}(\{1\}) \cap [0,x] = \{1,4\},$$
$$r_A^{-1}(\{2\}) \cap [0,x] = \{2\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \dots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 5$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0, x]\right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 6$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 7$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{|r_A^{-1}(\{t\}) \cap [0, x]|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 8$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 9$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 10$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0, x]\right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 11$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0, x]\right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 12$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9, 12\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 13$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9, 12\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10, 13\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 14$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11, 14\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9, 12\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10, 13\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$.
For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x\to\infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0,x]\right|}{x+1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 15$ we have

$$r_A^{-1}(\{0\}) \cap [0,x] = \{0, 3, 6, 8, 11, 14\},$$
$$r_A^{-1}(\{1\}) \cap [0,x] = \{1, 4, 7, 9, 12, 15\},$$
$$r_A^{-1}(\{2\}) \cap [0,x] = \{2, 5, 10, 13\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \dots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0, x]\right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 16$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11, 14, 16\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9, 12, 15\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10, 13\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \dots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x\to\infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0,x]\right|}{x+1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 17$ we have

$$r_A^{-1}(\{0\}) \cap [0,x] = \{0, 3, 6, 8, 11, 14, 16\},$$
$$r_A^{-1}(\{1\}) \cap [0,x] = \{1, 4, 7, 9, 12, 15, 17\},$$
$$r_A^{-1}(\{2\}) \cap [0,x] = \{2, 5, 10, 13\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \dots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0, x]\right|}{x+1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 18$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11, 14, 16, 18\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9, 12, 15, 17\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10, 13\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \dots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d} r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left| r_A^{-1}(\{t\}) \cap [0, x] \right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 19$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11, 14, 16, 18\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9, 12, 15, 17, 19\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10, 13\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^{N}$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \ldots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

$$\mathbf{d}r_A^{-1}(\{t\}) := \lim_{x \to \infty} \frac{\left|r_A^{-1}(\{t\}) \cap [0, x]\right|}{x + 1}$$

if the limit exists.

Illustration

Suppose $A = \{3, 8, 18, 21\}$. For $x = 20$ we have

$$r_A^{-1}(\{0\}) \cap [0, x] = \{0, 3, 6, 8, 11, 14, 16, 18\},$$
$$r_A^{-1}(\{1\}) \cap [0, x] = \{1, 4, 7, 9, 12, 15, 17, 19\},$$
$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10, 13, 20\}.$$

$r_A(x)$ for arbitrary A

Write $A = \{a_n\}_{n=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ and $a_n < a_{n+1}$ for $1 \leqslant n < N$. For every $t \in \{0, \dots, a_1 - 1\}$, define the density of $r_A^{-1}(\{t\})$ as

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$$r_A^{-1}(\{2\}) \cap [0, x] = \{2, 5, 10, 13, 20\}.$$

Fact

For every $x \in \mathbb{N}_0$ we have

$$\left| r^{-1}(\{0\}) \cap [0, x] \right| \geqslant \cdots \geqslant \left| r^{-1}(\{a_1 - 1\}) \cap [0, x] \right|.$$

$r_A(x)$ for arbitrary A

If $A \subseteq a_1\mathbb{N}$, then for every $t \in \{0, \dots, a_1 - 1\}$ and $x \in \mathbb{N}_0$ we have $r_A(x) = x \bmod a_1$, and so

$$\left|r_A^{-1}(\{t\}) \cap [0, x]\right| = |\{y \in \mathbb{N}_0 : y \bmod a_1 = t\} \cap [0, x]| = \left\lfloor \frac{x - t}{a_1} \right\rfloor + 1.$$

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It follows that for every $x \in \mathbb{N}_0$ we have

$$\frac{\left|r^{-1}(\{a_1 - 1\}) \cap [0,x]\right|}{x+1} \geqslant \frac{\lfloor (x+1)/a_1 \rfloor}{x+1} \xrightarrow{x\to\infty} \frac{1}{a_1}$$

and

$$\frac{\left|r^{-1}(\{0\}) \cap [0,x]\right|}{x+1} \leqslant \frac{\lfloor x/a_1 \rfloor + 1}{\lfloor x/a_1 \rfloor a_1 + 1} \xrightarrow{x\to\infty} \frac{1}{a_1}.$$

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If $|A| < \infty$ and $A \not\subseteq a_1\mathbb{N}$

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If $A \subseteq a_1\mathbb{N}$, then for every $t \in \{0, \dots, a_1 - 1\}$ and $x \in \mathbb{N}_0$ we have $r_A(x) = x \bmod a_1$, and so

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If $|A| < \infty$ and $A \not\subseteq a_1\mathbb{N}$ then for every $i \in \mathbb{N}$,

$$\left|r_A^{-1}(\{0\}) \cap [(i-1)a_N, ia_N - 1]\right| - \left|r_A^{-1}(\{a_1 - 1\}) \cap [(i-1)a_N, ia_N - 1]\right| \geqslant 1,$$

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If $A \subseteq a_1\mathbb{N}$, then for every $t \in \{0, \ldots, a_1 - 1\}$ and $x \in \mathbb{N}_0$ we have $r_A(x) = x \bmod a_1$, and so

$$\left|r_A^{-1}(\{t\}) \cap [0,x]\right| = |\{y \in \mathbb{N}_0 : y \bmod a_1 = t\} \cap [0,x]| = \left\lfloor \frac{x-t}{a_1} \right\rfloor + 1.$$

It follows that for every $x \in \mathbb{N}_0$ we have

$$\frac{\left|r^{-1}(\{a_1 - 1\}) \cap [0,x]\right|}{x+1} \geqslant \frac{\lfloor (x+1)/a_1 \rfloor}{x+1} \xrightarrow{x\to\infty} \frac{1}{a_1}$$

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If $|A| < \infty$ and $A \not\subseteq a_1\mathbb{N}$ then for every $i \in \mathbb{N}$,

$$\left|r_A^{-1}(\{0\}) \cap [(i-1)a_N, ia_N - 1]\right| - \left|r_A^{-1}(\{a_1 - 1\}) \cap [(i-1)a_N, ia_N - 1]\right| \geqslant 1,$$

and so

$$\left|r_A^{-1}(\{0\}) \cap [0, ia_N - 1]\right| - \left|r_A^{-1}(\{a_1 - 1\}) \cap [0, ia_N - 1]\right| \geqslant i.$$

$r_A(x)$ for arbitrary A

Theorem

If $A \subseteq a_1\mathbb{N}$, then the density sequence $\left(\mathbf{d} r_A^{-1}(\{t\})\right)_{t=0}^{a_1-1}$ exists and is constant. The converse also holds in the case $|A| < \infty$.

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Examples

- If $A = \{6n-3\}_{n=1}^{\infty} \subseteq 3\mathbb{N}$, then the density sequence is constant.

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Examples

- If $A = \{6n-3\}_{n=1}^{\infty} \subseteq 3\mathbb{N}$, then the density sequence is constant.
- If $A = \{6n-3\}_{n=1}^{\infty} \cup B$, where $\varnothing \neq B \subseteq \{6n-4\}_{n=2}^{\infty} \cup \{6n-5\}_{n=2}^{\infty}$ and $|B \cap [0,x]| = o(x)$, then the density sequence remains constant although $A \not\subseteq 3\mathbb{N}$.

$r_A(x)$ for arbitrary A

Theorem

If $A \subseteq a_1\mathbb{N}$, then the density sequence $\left(\mathbf{d}r_A^{-1}(\{t\})\right)_{t=0}^{a_1-1}$ exists and is constant. The converse also holds in the case $|A| < \infty$.

Examples

- If $A = \{6n-3\}_{n=1}^{\infty} \subseteq 3\mathbb{N}$, then the density sequence is constant.
- If $A = \{6n-3\}_{n=1}^{\infty} \cup B$, where $\varnothing \neq B \subseteq \{6n-4\}_{n=2}^{\infty} \cup \{6n-5\}_{n=2}^{\infty}$ and $|B \cap [0,x]| = o(x)$, then the density sequence remains constant although $A \not\subseteq 3\mathbb{N}$.
- If $A = \{6n-3\}_{n=1}^{\infty} \cup B$, where $B = \{4\}$, then

$$r_A^{-1}(\{t\}) = \begin{cases} \{0,4,8\} \cup (6\mathbb{N}-3) \cup (6\mathbb{N}+1) \cup (6\mathbb{N}+6), & \text{if } t = 0; \\ \{1,5\} \cup (6\mathbb{N}+4) \cup (6\mathbb{N}+8), & \text{if } t = 1; \\ \{2,6\} \cup (6\mathbb{N}+5), & \text{if } t = 2, \end{cases}$$

and so the density sequence is not constant.

$r_A(x)$ for arbitrary A

Suppose $A = \{a'_n + (n-1)d\}_{n=1}^{\infty}$, where $2 \leqslant a'_1 \leqslant a'_2 \leqslant \cdots$ and $d \in \mathbb{N}_0$.

$r_A(x)$ for arbitrary A

Suppose $A = \{a'_n + (n-1)d\}_{n=1}^{\infty}$, where $2 \leqslant a'_1 \leqslant a'_2 \leqslant \cdots$ and $d \in \mathbb{N}_0$.

Then for every $x_1 \in \left[a'_n + (n-1)d, a'_{n+1} + nd\right)$, we have

$$x_2 = x_1 - [a'_n + (n-1)d] < \left(a'_{n+1} - a'_n\right) + d.$$

$r_A(x)$ for arbitrary A

Suppose $A = \{a'_n + (n-1)d\}_{n=1}^{\infty}$, where $2 \leqslant a'_1 \leqslant a'_2 \leqslant \cdots$ and $d \in \mathbb{N}_0$.

Then for every $x_1 \in \left[a'_n + (n-1)d, a'_{n+1} + nd\right)$, we have

$$x_2 = x_1 - [a'_n + (n-1)d] < \left(a'_{n+1} - a'_n\right) + d.$$

If $a'_{n+1} - a'_n < a'_2$, then

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$r_A(x)$ for arbitrary A

Suppose $A = \{a'_n + (n-1)d\}_{n=1}^{\infty}$, where $2 \leqslant a'_1 \leqslant a'_2 \leqslant \cdots$ and $d \in \mathbb{N}_0$.

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If $a'_1 \mid a'_n$ for every $n \in \mathbb{N}$, then

$$\left| r_A^{-1}(\{0\}) \cap \left[0, a'_{n+1} + nd\right) \right| \sim \frac{a'_{n+1}}{a'_1} + n \left\lceil \frac{d}{a'_1} \right\rceil,$$

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$$\mathbf{d} r_A^{-1}(\{0\}) = \lim_{n\to\infty} \frac{a'_{n+1} + na'_1 \lceil d/a'_1 \rceil}{a'_1 \left(a'_{n+1} + nd\right)}.$$

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Theorem

Let $d \in \mathbb{N}_0$. Let $A = \{a'_n + (n-1)d\}_{n=1}^{\infty} \subseteq \mathbb{N}$, where $2 \leqslant a'_1 \leqslant a'_2 \leqslant \cdots$ and $a'_1 \mid a'_n$ for every $n \in \mathbb{N}$. If there exists $m \in \mathbb{N}$ such that for every integer $n \geqslant m$ we have $a'_{n+1} - a'_n < a'_2$, then

$$\mathbf{d} r_A^{-1}(\{0\}) = \lim_{n\to\infty} \frac{a'_{n+1} + n a'_1 \lceil d/a'_1 \rceil}{a'_1 \left(a'_{n+1} + nd\right)},$$

provided the limit exists.

$r_A(x)$ for arbitrary A

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provided the limit exists.

Example

- If $A = \{a_1 + (n-1)d\}_{n=1}^{\infty} \subseteq \mathbb{N}$ is an arithmetic progression, where $d \in \mathbb{N}$, then

$$\mathbf{d} r_A^{-1}(\{0\}) = \frac{1}{d} \left\lceil \frac{d}{a_1} \right\rceil.$$

  This density is equal to $1/a_1$ if and only if $a_1 \mid d$, and is equal to 1 if and only if $a_1 = 1$ or $d = 1$.

$r_A(x)$ for arbitrary A

Example

- Let $A = \{2\} \cup 7\mathbb{N}$. Then $A = \{a'_n + (n-1)d\}_{n=1}^{\infty}$, where $d = 5$ and

$$a'_1 = 2 \qquad \text{and} \qquad a'_n = 2n - 2 \quad \text{for every } n \geqslant 2.$$

In particular, we have $a'_{n+1} - a'_n = 2$ for every $n \geqslant 2$, and hence the required integer m does not exist.

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In particular, we have $a'_{n+1} - a'_n = 2$ for every $n \geqslant 2$, and hence the required integer m does not exist. As a result, while

$$\lim_{n\to\infty} \frac{a'_{n+1} + na'_1 \lceil d/a'_1 \rceil}{a'_1 \left(a'_{n+1} + nd\right)} = \frac{1}{2},$$

we have

$$r_A^{-1}(\{0\}) = \{0, 2, 4, 6\} \cup 7\mathbb{N} \cup (7\mathbb{N} + 2) \cup (7\mathbb{N} + 4) \cup (7\mathbb{N} + 6),$$

and so

$$\mathbf{d} r_A^{-1}(\{0\}) = \frac{4}{7}.$$

References

- J. Hoseana and Steven, On a sequence of densities generated by the greedy algorithm, São Paulo J. Math. Sci. (2022).
- F. Luca and R. Thangadurai, On an arithmetic function considered by Pillai, J. Théor. Nr. Bordx. 21 (2009), 693–699.
- A. Mukhopadhyay, R. Thangadurai, and G. K. Viswanadham, Unique representation of integers with base *A*, Arch. Math. 105 (2015), 119–128.
- S. S. Pillai, On an arithmetic function concerning primes, Annamalai Univer. J. 1 (1930), 159–167.
- Steven and J. Hoseana, The prime-power map, J. Integer Seq. 24 (2021), 21.2.2.

Jonathan Hoseana
j.hoseana@unpar.ac.id
sites.google.com/view/jonathanhoseana